A proposal for analyzing the classical limit of kinematic loop gravity.

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ABSTRACT

We analyze the classical limit of kinematic loop quantum gravity in which the diffeomorphism and hamiltonian constraints are ignored. We show that there are no quantum states in which the primary variables of the loop approach, namely the SU(2) holonomies along *all* possible loops, approximate their classical counterparts. At most a countable number of loops must be specified. To preserve spatial covariance, we choose this set of loops to be based on physical lattices specified by the quasi-classical states themselves. We construct "macroscopic" operators based on such lattices and propose that these operators be used to analyze the classical limit. Thus, our aim is to approximate classical data using states in which appropriate macroscopic operators have low quantum fluctuations.

Although, in principle, the holonomies of 'large' loops on these lattices could be used to analyze the classical limit, we argue that it may be simpler to base the analysis on an alternate set of "flux" based operators. We explicitly construct candidate quasi-classical states in 2 spatial dimensions and indicate how these constructions may generalize to 3d. We discuss the less robust aspects of our proposal with a view towards possible modifications. Finally, we show that our proposal also applies to the diffeomorphism invariant Rovelli model which couples a matter reference system to the Hussain Kuchař model.

1. Introduction

The loop quantum gravity approach has yielded a number of interesting results. A mathematical arena has been defined in which the constraints of quantum gravity have been expressed as quantum operators. The complete kernel of the diffeomorphism constraints has been obtained [1] and efforts are on to find the kernel of the Hamiltonian constraint [2, 3] However, contact with the classical limit (i.e. general relativity) has been elusive.

Since very little is known regarding the interpretation of the kernel of the constraint operators, the unambiguous results pertaining to the classical limit have been obtained at the kinematic level wherein the diffeomorphism and Hamiltonian constraints are ignored [4, 5, 6]. By taking recourse to the arguments of Rovelli [7, 8], it is, however, not inconceivable that kinematic results may be physically relevant. Moreover, in any situation with classical boundary conditions (e.g. black hole horizons, asymptotically flat spacetimes), the classical constraint vector fields leave the boundary conditions invariant. Hence, at the boundary, the smearing functions (lapse and shift) for the constraints typically vanish and kinematic results may acquire physical significance.

Even at the kinematic level, almost all work to date is restricted to an exploration of the classical limit of (functionals of) the spatial metric or densitized triad operators [6, 9]. [1] 'Weave' states have been constructed which approximate classical metrical information. It is possible that the conjugate (connection) variable fluctuates wildly in such states and, if so, these states cannot be quasi-classical.

In this work we propose a framework to analyze both the metrical as well as the connection degrees of freedom with a view towards the classical limit. The reason a new framework is required is as follows.

The connection dependent operators which have unambiguous classical counterparts are the traces of holonomies around loops. The latter are denoted by $T^0_\gamma(A)$ with

$$T^0_\gamma(A) = \frac{1}{2} Tr H_\gamma(A), \quad H_\gamma(A) = P \exp - \oint_\gamma A_a dx^a, \tag{1}$$

where γ is a loop embedded in the spatial manifold Σ, A_a is an SU(2) connection, $H(\gamma)$ is the holonomy and Tr denotes the trace in the $j = \frac{1}{2}$ representation. It would be natural to explore the classical limit in terms of these operators. Then, quasiclassical states would be required to approximate the set of all holonomies and say, surface areas, through quantum expectation values of the corresponding operators with low fluctuations. Unfortunately, as we show in the beginning of section 2, holonomies of a classical connection along all possible loops cannot be approximated by any quantum state! More precisely, it is only on a countable set of loops (the set of all loops is, of course, uncountable) that holonomies have a chance of being approximated. However, an arbitrary choice of this countable set is in conflict with spatial covariance.

Therefore, a new framework is needed to analyze the connection degrees of freedom. In this work we propose such a framework. Our main ideas are as follows. Any

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[1] An exception is [10].

quasiclassical state must approximate, in some way, the data corresponding to both the spatial metric as well as the connection. To approximate a given spatial metric we need states defined on a "large enough" graph. We require that this graph gives a latticization of the (compact) spatial manifold. Then the preferred set of loops are naturally identified as those which lie on the lattice. We make these ideas precise in section 2 in such a way that the resulting framework is spatially covariant.

Next, given the set of loops on a lattice, we would like to approximate a classical connection. The natural set of operators to consider are the holonomies along these loops. Since we are interested in approximating classical behaviour at scales much larger than the Planck length, it is enough to restrict attention to loops of size much larger than the Planck scale (the size of a loop is measured by the metric part of the classical data). Thus one natural set of connection operators for an analysis of the classical limit are the holonomies along large loops which lie on the lattice. However, the lattice structure suggests an alternative set of operators. These are the 'magnetic flux' operators of lattice gauge theory which measure the non-abelian magnetic flux through the plaquettes of the lattice. They are constructed in the usual way from holonomies along the plaquettes. For reasons which we spell out in section 3, we choose to base our analysis of the classical limit on these operators rather than the large loop holonomies. We devote section 3 to this change of focus from holonomy operators to flux operators.

In section 4 we work out our ideas in detail for the case of two spatial dimensions and explicitly display states which approximate aspects of both the classical spatial metric and the $SU(2)$ connection. We also indicate how our constructions can be extended to the case of three spatial dimensions.

Section 5 is devoted to a discussion of various issues which arise in the context of our proposal, with an emphasis on its less robust aspects. The discussion in this section indicates that some of our ideas are too simplistic whereas others posess attractive features; it thus points to ways in which the proposal may be modified. We also show, in section 5, how our proposal can be extended to the diffeomorphism invariant context of Rovelli's work [8] wherein the Hussain Kuchař model is coupled to a matter reference system. Section 6 contains our conclusions.

There seems to be no single viewpoint with regard to the role, within the framework of loop quantum gravity, of considerations at the purely kinematic level. Therefore it is appropriate that we spell out our viewpoint before describing our results.

The aim of loop quantum gravity is to construct a quantum theory which has general relativity as its classical limit. Since, in this approach, the Hamiltonian constraint operator is poorly understood, it would be premature to discuss the classical limit at the full dynamical level. Even at the (spatial) diffeomorphism invariant level, with the exception of the total volume operator, quantum operators corresponding to diffeomorphism invariant classical observables have not been constructed. Without these operators it is difficult to interpret the theory and discuss its classical limit. Since even the kinematic state space is very different from that of conventional flat space quantum field theory, it makes sense to understand the classical limit first at this kinematic level, where even the diffeomorphism constraints are ignored. The classical limit consists of smooth metric and connection data. The approximation

of smooth metrical data by weave states is already subtle and the approximation of smooth connection data is still an open question. It is our view that an analysis of the classical limit at the kinematic level may clarify strategies for analysing the classical limit at the spatial diffeomorphism invariant level and finally, (once the Hamiltonian constraint is well understood) at the fully dynamical level.

Independent of the above 'structural' role of understanding the classical limit of kinematic gravity, is the question of whether results at the kinematic level have any relevance to *physical* predictions of full blown quantum gravity. [2] For example, the discreteness of the spectrum of the area operator is often cited by some workers as a physical prediction. Since the Hamiltonian constraint is not well understood, we refrain from discussing this issue in the context of full quantum gravity. Instead, we restrict our attention to the possibility of promoting kinematic results to predictions at the diffeomorphism invariant level. One way to promote kinematic results to the diffeomorphism invariant level, is to couple the gravitational variables to a matter reference system as in, for example, [8]. This can be done only if the kinematic framework for the gravitational variables is spatially covariant. In what follows, we shall be guided by this requirement of spatial covariance.

Notation and Conventions: We assume familiarity with the loop quantum gravity approach (for example see [1] and references therein) and use notation which is standard in the field. $a, b..$ are spatial indices, $i, j..$ are internal $SU(2)$ indices, $A_a^i(x)$ is the $SU(2)$ connection and $\tilde{E}_i^a(x)$ is the densitized triad.

$\{A_a^i(x), \tilde{E}_i^b(y)\} = \iota G_0 \delta_a^b \delta(x, y)$ where ι is the (real) Immirzi parameter [11]. We shall restrict attention to piecewise analytic loops/graphs.

$\overline{\mathcal{A}}$ is the completion (via a projective limit construction) of the space of smooth connections $\mathcal{A}$, $\overline{\mathcal{A}/\mathcal{G}}$ is the Gel'fand completion of the space of smooth connections modulo gauge and $d\mu_H$ denotes the Ashtekar-Lewandowski (or Haar) measure on $\overline{\mathcal{A}}$ as well as on $\overline{\mathcal{A}/\mathcal{G}}$.

$\hat{O}$ is the operator version of the classical object O, $\hat{O}^\dagger$ is its adjoint and O^* is the complex conjugate of O. We shall often denote the expectation value of $\hat{O}$ in the quantum state under discussion as $< \hat{O} >$. l_{0P} is the length constructed from the dimension-full gravitational coupling G_0, $\hbar$ and c. In 3+1 dimensions, $l_{0P} = \sqrt{\frac{G_0 \hbar}{c^3}}$. We shall use units in which $\hbar = c = 1$.

2. The necessity for a new framework and a sketch of our proposal

The most straightforward approach to an analysis of the classical limit of loop quantum gravity would be to construct minimum uncertainty states for the basic operators of the theory. These operators are the 'configuration' operators, $\hat{T}_\gamma^0$, and suitable 'momentum' operators. The latter may be chosen as the area operators, $\hat{A}_S$ [4, 5] (A_S is the area of a surface S in Σ).

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[2]Note that we are not considering those special situations mentioned earlier in this section, involving boundary conditions, where kinematic results are already 'gauge invariant'.

A tentative definition of a quasi-classical state as a minimum uncertainty state for this set of operators is as follows. A kinematic quasi-classical state $|\psi>\in L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ which approximates the $SU(2)$ gauge equivalence class of the classical data, $(A^i_{0a}(x), \tilde{E}^b_{0i}(x))$, is such that, for all γ, S

(i) $|<\hat{T}^0_\gamma> -T^0_\gamma(A_0)|$ and $\Delta\hat{T}^0_\gamma = (<(\hat{T}^0_\gamma)^2> - <\hat{T}^0_\gamma>^2)^{\frac{1}{2}}$ are small.

(ii) $|<\hat{A}_S> -A_S(E_0)|$ and $\Delta\hat{A}_S$ are small compared to $A_S(E_0)$.

Since $|T^0_\gamma(A)| < 1$, we interpret 'small' in (i) as 'small compared to 1'.

We now show that no quantum state exists in the kinematic Hilbert space for which (i) is true for all loops γ. The kinematical Hilbert space, $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$, is spanned by the set of cylindrical functions, each of which is labelled by a piecewise analytic, closed, finite graph. Hence any element of the Hilbert space is associated with at most a countable infinite set of closed graphs. From the properties of $d\mu_H$ it is easy to see that given any such state $|\psi>\in L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$, and any loop α which does not belong to the countable set of graphs associated with $|\psi>$, $<\psi|\hat{T}^0_\alpha|\psi> = 0$ and $\Delta\hat{T}^0_\alpha = \frac{1}{2}$. Clearly, there are uncountably many loops of the type α. It follows that there is no state for which (i) holds for all loops γ in Σ.

Hence, the most straightforward approach to an analysis of the connection degrees of freedom fails and a different approach, which relaxes (i) in some way, needs to be formulated. The remainder of this section is devoted to the construction of such an approach.

¿From our arguments above, it is clear that we must relax (i) to hold for at most a countable set of loops. It is reasonable to require that the structure of this set of loops be such that we can use them to approximate, in some way, *any* given loop. A lattice structure is one which has this property. Thus, we are naturally led to require that the set of loops for which (i) is imposed provides a latticization of the compact spatial manifold. (For simplicity, we shall restrict attention to lattices with a finite, though arbitrarily large, number of links).

However, an arbitrary fixed choice of such a lattice (or indeed, of any other countable set of loops) introduces a preferred structure into the description and hence breaks spatial covariance.[3] We get around this difficulty as follows.

It is essential to note that we are only interested in quantum states which approximate classical data. In particular such states approximate the data for the classical spatial metric. States which approximate *only* the spatial metric have been constructed in [6, 12] and are based on an underlying graph. If this graph does not extend into a region $R \subset \Sigma$ then R has zero volume and any surface in R has zero area. Hence a state based on such a graph does not correspond to any classical metric in the region R. It follows that the graph underlying a quasi-classical state must 'extend into all of Σ' in order to approximate a classical metric on Σ. Such graphs are called weaves [6, 12]. For our purposes it seems natural to require that the graph underlying any weave state which not only approximates a classical 3- metric, but also approximates a classical connection (modulo $SU(2)$ gauge), provides a latticization

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[3]Diffeomorphisms are unitarily represented on the kinematic Hilbert space. Spatial covariance implies that classical data sets differing by the action of diffeomorphisms are approximated by quantum states which differ by the action of the corresponding unitary operators.

of Σ. More precisely, we require that the graph be the 1-skeleton of some cellular complex [13] whose topology is that of Σ. [4] Thus, the required lattice structure is not chosen arbitrarily but is obtained from the quasiclassical state itself. It is this feature which preserves the spatial covariance of the resulting framework.

The availability of a lattice structure enables us to analyse many more functions than just the holonomies. More precisely, any function which admits a lattice approximant may be analysed using techniques from lattice gauge theory. Therefore, we shall develop our framework in such a way as to deal with any function of the classical data which admits a lattice approximant (the degree of approximation will be made quantitative shortly).

To make these ideas more precise, we define the following mathematical structures. Let L denote a finite piecewise analytic graph which provides a latticization of the compact manifold Σ. Note that L belongs to an uncountably infinite label set, since the action of a diffeomorphism on L produces a lattice L' which is, in general, different from L.

We define the lattice projector $\hat{P}_L$ as the projection operator which maps any state in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H) \subset L^2(\overline{\mathcal{A}}, d\mu_H)$ to its component in the subspace spanned by spin network states [17, 18] which have the following properties: (a) every spin network state in the subspace is labelled by the graph L, and (b) for every such spin network state, every link of the graph L is labelled by some non-trivial (i.e. $j \neq 0$) representation of $SU(2)$. It can be checked that

$$\hat{P}_L \hat{P}_{L'} = \delta_{L,L'} \hat{P}_L \tag{2}$$

where $\delta_{L,L'} = 0$ if $L \neq L'$ and $\delta_{L,L'} = 1$ if $L = L'$. Also $\hat{P}_L$ is a (bounded) self adjoint operator on $L^2(\overline{\mathcal{A}/\mathcal{G}})$ so that

$$\hat{P}_L = \hat{P}_L^\dagger. \tag{3}$$

Denote the space of finite linear combinations of spin networks associated with all the graphs contained in the graph L by $\mathcal{D}_L$ and its completion in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ as $\mathcal{H}_L$.[5] Note that $\mathcal{H}_L$ is the Hilbert space of $SU(2)$ lattice gauge theory on the lattice L.

Let $\hat{O}_L$ be a bounded self adjoint operator on $\mathcal{H}_L$ (or a densely defined symmetric operator on $\mathcal{D}_L$). Then define the operator $\hat{O}$ as

$$\hat{O} := \sum_L \hat{P}_L \hat{O}_L \hat{P}_L. \tag{4}$$

Here, the sum is over all possible latticizations of Σ. $\hat{O}$ has the following well defined action on any spin network state in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$. Every spin network state is associated with some unique 'coarsest' graph i.e. the graph which has all its edges labelled by non zero spin. Let γ_0 be the coarsest graph for the spin network state

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[4]Most of the weaves constructed in the literature (see [12] and references therein) are the disjoint union of sets of loops, and do not provide a latticization of Σ. Notable exceptions are the boundary data of spin foam models (see [14, 15, 16]).

[5]Note that, since all spin network states based on L (including those with some or all links labelled by $j = 0$) are contained in $\mathcal{H}_L$, $\mathcal{H}_L \neq \hat{P}_L(L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H))$.

ψ_{γ_0}. Then, if γ_0 does not provide a latticization of Σ, from (4), $\hat{O}\psi_{\gamma_0} = 0$ otherwise $\hat{O}\psi_{\gamma_0} = \hat{P}_{\gamma_0}\hat{O}_{\gamma_0}\psi_{\gamma_0}$. This action can be extended by linearity to the dense set of finite linear combinations of spin network states in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ and thus $\hat{O}$ is a densely defined operator on this dense domain.

We now use (4) to encode our ideas for the approximation of classical data $(A^i_{0a}(x), \tilde{E}^b_{0i}(x))$. Let the classical metric constructed from $\tilde{E}^a_{0i}(x)$ be q_{oab}. Let O_L be the classical lattice approximant to the (real) classical quantity O on the lattice L. Typically, for classical functions of interest, the lattice function O_L is a sum over the 'cell' functions O_{I_L} where I_L labels the cells/plaquettes of the lattice L. The finer the lattice L, the closer is O_L to the continuum function O and the larger is the number of 'cell' contributions to O_L. The degree to which O_L approximates O can be made quantitative in terms of the length of the lattice parameters of L as measured by q_{oab}.

Let $\hat{O}_L$ be the operator corresponding to O_L. We require that $\hat{O}_L$ be constructed as a self adjoint operator on $\mathcal{H}_L$ (or $\mathcal{D}_L$), from magnetic flux type operators of $SU(2)$ lattice gauge theory on the lattice L. Then, *for calculations of expectation values in a quasi-classical state* we interpret (4) as the operator corresponding to the classical quantity O. Recall that we require quasi-classical states to be associated with some lattice L. From the considerations of [6], it is expected that the typical link size of such a lattice as measured by q_{oab} is of the order of the Planck length. Thus, the only term to contribute to an expectation value in a quasi-classical state in the right hand side of (4), will be one associated with a lattice with Planck size lattice parameters!

This completes the description of our proposed framework but for one last issue. Since the operator $\hat{O}$ has the lattice projection operators, P_L, in its definition, it is not obvious that the usual correspondence is guaranteed between the Poisson brackets of macroscopic classical quantities O and the commutators of the corresponding operators $\hat{O}$. Thus it must be checked if this correspondence holds in expectation value in order that our candidate quasi-classical states be physically acceptable.

We can now summarize our proposed framework for analysing quasiclassicality as follows:

(1) We require that any quasi-classical state ψ, which approximates both the classical 3- metric and the conjugate connection, (A_0, E_0), be associated with some lattice L_0, so that $\hat{P}_{L_0}\psi = \psi$. [6]

(2) Given a classical function, O, we construct a corresponding operator $\hat{O}$ as follows. We identify the lattice approximant O_L to O and construct the operator $\hat{O}_L$ in the lattice gauge theory on L. Then we construct $\hat{O}$ as in (4).

(3) We require that $< O >\sim O(A_0, E_0)$, that $\Delta\hat{O}$ be small compared to typical classical values of the function O and that the usual correspondence between commutators and Poisson brackets holds for expectation values in quasi-classical states.

We end this section with a few technical remarks. If for every L, $\hat{O}_L$ is a bounded self adjoint operator on $\mathcal{H}_L$ then using Lemma 1, section 4.4 of [19], it can be verified that $\hat{O}$ is an essentially self adjoint operator on the dense domain of finite linear combinations of spin networks in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$.

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[6]Though we shall not do so here, it seems natural to relax this condition and only require (2) and (3) of any quasiclassical state.

However, typically, the operators $\hat{O}_L$ of interest are (unbounded) densely defined symmetric operators on $\mathcal{D}_L$. Then it is straightforward to see that $\hat{O}$ is a densely defined symmetric operator on the dense domain of finite linear combinations of spin network states in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$.

3. 'Magnetic flux' operators.

Holonomies serve as natural candidates for the classical functions 'O' of section 2, in as much as the connection degrees of freedom are concerned. ¿From the considerations of section 2, we restrict attention to holonomies along loops which lie on the lattice associated with a quasiclassical state. The classical metric being approximated endows every such loop with a size. Clearly, it does not make sense to require that holonomies along Planck size loops display classical behaviour; it is only for loops of size much larger than the Planck scale, that we expect classical behaviour. Hence, we may further restrict our attention to holonomies along such "large" loops.

A different set of operators than the large loop holonomies is suggested by the lattice structure. These operators are the magnetic flux operators of lattice gauge theory which measure the non-abelian magnetic flux through the plaquettes of the lattice. They are defined in the natural way via holonomies along the plaquettes [20].

Since a single plaquette is typically of Planck size, we shall refer to the magnetic flux through a plaquette as the 'microscopic' magnetic flux. Clearly, the microscopic magnetic flux is not of direct relevance to the classical limit. It is only 'macroscopic' operators associate with 'macroscopic' length scales (i.e. length scales far above the Planck scale) that are relevant to the classical limit. The utility of the microscopic magnetic flux (or equivalently, the holonomy along a 'microscopic' loop) is that it serves as the lattice approximant to the curvature of the connection - the curvature is approximated on the lattice by the flux through a plaquette divided by the plaquette area. Many physically interesting functions can be constructed from the curvature (for e.g. $D(\vec{N}) = \int_\Sigma N^a \tilde{E}_i^a F_{ab}^i$, where N^a is a vector field and F_{ab}^i the curvature of the connection) and thus, admit lattice approximants built out of microscopic fluxes.

It turns out, as we show in section 3.1, that because of the differences in their algebraic properties, it is simpler to use the flux operators rather than the holonomies along macroscopic loops, to analyse the classical limit. Moreover, as discussed in section 3.2, the consideration of flux-based macroscopic operators suggests a general strategy to build states in which these operators have low relative fluctuations. For these reasons we shift focus from the holonomies of macroscopic loops to flux based macroscopic operators in our explicit constructions of section 4. As we shall see in section 5, the strategy discussed in section 3.2 is not entirely successful; nevertheless this strategy springs from an interesting idea and, among other things, this work is devoted to examining it in detail.

3.1. Algebraic properties of holonomies vs fluxes.

The holonomies and fluxes have very different algebraic properties. Fluxes are associated with 2d surfaces and and are *additive*. The flux through the union, S, of disjoint surfaces S_I, $I = 1..M$ is the sum of the fluxes through each of the surfaces,

$$\int_S F_{ab}^i = \sum_{I=1}^M \int_{S_I} F_{ab}^i. \tag{5}$$

Here F_{ab}^i is the curvature of the connection pulled back to the relevant 2 surface and i is some fixed internal $SU(2)$ direction. Equivalently, defining the flux $\Phi^i(S) = \int_S F_{ab}^i$,

$$\Phi^i(S) = \sum_{I=1}^M \Phi^i(S_I) \tag{6}$$

In contrast holonomies are associated with 1d loops and are *multiplicative*. Thus if $\gamma := \gamma_1 \circ \gamma_2.... \circ \gamma_N$ is the loop composed of the loops $\gamma_I, I = 1..N$,

$$H_\gamma(A) = \prod_{I=1}^N H_{\gamma_I}(A) \tag{7}$$

where the product signifies group multiplication.

Thus, (6) determines the flux through large surfaces in terms of small surfaces which combine to form the large surfaces and (7) determines the holonomy of a composite loop in terms of the holonomies of the loops which compose it.

By definition, (6) also holds for the quantum flux operators and hence for their expectation values. Thus, the expectation values of the fluxes through small surfaces determine the expectation value of the flux through the large surface via the quantum version of (6). This simplifies the construction of quasiclassical states since it suffices to restrict attention to a smaller "basis" set of surfaces from which all surfaces of interest can be composed. Similar considerations hold for gauge invariant flux based macroscopic operators.

In contrast, although (7) also holds for the holonomy operators, it does not necessarily hold for their expectation values due to quantum fluctuations. In fact, as we show below, if (7) is *imposed* as a relation between expectation values in a quantum state, that state cannot be quasiclassical. This complicates the construction of quasiclassical states; since we cannot restrict attention to a smaller "basis" set of loops, the holonomies have to be approximated all at once. It is in this sense that it is easier to use fluxes than holonomies.

We now prove our claim regarding the holonomy expectation values.

On $L^2(\overline{\mathcal{A}}, d\mu_H)$ define the bounded self adjoint operators

$$\hat{x}_\alpha^0 := \hat{T}_\alpha^0, \qquad \hat{x}_\alpha^j := \frac{i}{2} Tr(\hat{H}_\alpha \sigma^j), \tag{8}$$

where σ^j are the 2×2 Pauli matrices. Since $H_\alpha(A) \in SU(2)$,

$$\sum_{\mu=0}^3 (\hat{x}^\mu)^2 = det\hat{H}_\alpha = 1. \tag{9}$$

For any state in $L^2(\overline{\mathcal{A}}, d\mu_H)$,

$$det < \hat{H}_\alpha >= \sum_{\mu=0}^{3} < \hat{x}^\mu >^2 . \tag{10}$$

¿From (9)

$$det < \hat{H}_\alpha >= 1 - \sum_{\mu=0}^{3} (\Delta \hat{x}_\alpha^\mu)^2. \tag{11}$$

¿From (10) and (11),

$$0 \leq det < \hat{H}_\alpha >\leq 1 - (\Delta \hat{T}_\alpha^0)^2 \tag{12}$$

Let $\gamma_I, I = 1..N$ be a set of N loops such that their composition is the loop γ. Thus, $\gamma := \gamma_1 \circ \gamma_2.... \circ \gamma_N$.

Let $(A_{0a}^i, \tilde{E}_{0i}^a)$ be the classical data to be approximated. Let $\epsilon > 0$ be a physically reasonable lower bound on the attainable uncertainty in the measurement of the $\hat{T}_{\gamma_I}^0$, $I = 1..N$. Thus,

$$\Delta \hat{T}_{\gamma_I}^0 \geq \epsilon > 0, \tag{13}$$

Since $H_\gamma(A_0) = \prod_{I=1}^{N} H_{\gamma_I}(A_0)$ we impose that

$$< \hat{H}_\gamma >\approx \prod_{I=1}^{N} < \hat{H}_{\gamma_I} > . \tag{14}$$

$$\Rightarrow det < \hat{H}_\gamma >\approx \prod_{I=1}^{N} det < \hat{H}_{\gamma_I} > . \tag{15}$$

Since $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H) \subset L^2(\overline{\mathcal{A}}, d\mu_H)$, we can use (12) to get

$$det < \hat{H}_\gamma >\leq \prod_{I=1}^{N} 1 - (\Delta \hat{T}_{\gamma_I}^0)^2. \tag{16}$$

From (10), (13) and (16)

$$| < \hat{T}_\gamma^0 > |^2 < det < \hat{H}_\gamma >< (1 - \epsilon)^N. \tag{17}$$

Since ϵ is independent of N, clearly, for sufficiently large N, the above equation implies that $| < \hat{T}_\gamma^0 > | << 1$. For generic A_{0a}^i there is no reason for the classical variable $T_\gamma^0(A_0)$ to be small. So if we assume that the classical connection of interest is such that

$$T_\gamma^0(A_0) \sim O(1), \tag{18}$$

then (i) is clearly violated for the loop γ because $| < \hat{T}_\gamma^0 > -T_\gamma^0(A_0)|$ is *not* much less than unity.

For loops of macroscopic size, we obtain rough estimates for N and ϵ as follows. Quantum gravitational fluctuations are not expected to be significant well above the Planck scale. So for the purposes of the gravitational interaction alone, energy scales of up to a few hundred Gev (or equivalently length scales larger than $10^{-16} cm$) can

safely be considered as 'classical'. A macroscopic size surface of the order of $100m^2$ contains the loops $\gamma_I, I = 1..N$, where each γ_I encloses a 'classical' size area of the order of $10^{-32}cm^2$. Thus N is of the order of 10^{38}. Even if ϵ is chosen as small as 10^{-34}, we obtain

$$| < \hat{T}^0_\gamma > | \leq (1 - 10^{-34})^{10^{38}} \sim e^{-10^4} \sim 0, \tag{19}$$

which clearly violates (i) for classical connections which satisfy (18)!

One way of arriving at a physically motivated choice for ϵ is as follows. In addition to the loops γ_I, consider a set of surfaces S_J, $J = 1..N$, each of classical size $10^{-32}cm^2$ such that each γ_I transversely intersects S_I exactly once. Then, choosing the orientation of S_I to be in the direction of γ_I, and denoting the area of the surface S_I by A_{S_I}, we have

$$\{T^0_{\gamma_I}, A_{S_I}\} = -\iota G_0 \frac{i}{2} Tr(H_{\gamma_I}\sigma^i)n_i, \tag{20}$$

where the right hand side is evaluated at the point of intersection between the loop γ_I and the surface S_I. n_i is defined as follows. Let $E^a_i := \frac{\tilde{E}^a_i}{\sqrt{q}}$ where q is the determinant of the metric constructed from the triad. Let n_a be the unit normal to the surface S_I defined by this metric. Then $n_i := n_a E^a_i$. Thus $n_i n^i = 1$ and we expect that for a large class of connections (with less than Planck scale curvature and which also satisfy (18)) and triads, it should be true that

$$\{T^0_{\gamma_I}, A_{S_I}\} = -\iota G_0 \frac{i}{2} Tr(H_{\gamma_I}\sigma^i)n_i \sim \iota G_0 O(1). \tag{21}$$

Note that if the above equation holds, then $Tr(H_{\gamma_I}\sigma^i)$ is of order unity. This implies that the curvature of the connection, F^i_{ab}, in physically reasonable coordinates is of the order of $10^{32}cm^{-2}$ which is still, for purposes of quantum gravity, classical.

For quasi-classical states we expect that the Poisson bracket to quantum commutator correspondence holds in the sense of expectation value so that

$$i\hbar\{T^0_\gamma, A_{S_I}\} \sim < [\hat{T}^0_\gamma, \hat{A}_{S_I}] > . \tag{22}$$

Combining (21) with (22) with the uncertainty principle for $\Delta\hat{T}^0_{\gamma_I}$, $\Delta\hat{A}_{S_I}$ we get

$$\Delta\hat{A}_{S_I}\Delta\hat{T}^0_{\gamma_I} \sim \iota l^2_{0P} \tag{23}$$

Let us assume, to be conservative, a huge uncertainty in the measurement of area [7] equal to $10^{-32}cm^2$ and set ιl^2_{0P} to be of the order of the Planck area (the latter is consistent with the black hole entropy calculations of [21]). Then from (23) $\epsilon = \frac{10^{-66}}{10^{-32}} = 10^{-34}$.

Finally, we note that (14) mirrors the relations (7) between classical holonomies. Since classical holonomies are not gauge invariant objects, it is necessary to extend

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[7]Note that our estimates are in the context of a thought experiment in which the only quantum effects are from the gravitational interaction. In practice, it would of course be almost impossible to directly make the appropriate measurements, due, in part, to the quantum nature of any interaction used in the measuring process.

our arguments to the gauge invariant context of traces of holonomies. We do this in appendix A1 by using Giles' (re)construction [22] of holonomies from their traces.

This completes our discussion as to why it is technically simpler to use fluxes as opposed to holonomies.

3.2. A general strategy for low fluctuations based on flux operators

In this section we describe a general strategy to obtain low relative fluctuations of flux based 'macroscopic' operators. This strategy is patterned on the mechanism for low relative fluctuations in statistical mechanics. In the statistical mechanics description of thermodynamic systems, there are 'N' weakly correlated degrees of freedom, N being very large. Mean values of macroscopic quantities typically go as N times some microscopic quantity whereas the relative fluctuations about the mean go as $\frac{1}{\sqrt{N}}$. It is the poor correlation between the degrees of freedom that is responsible for such low relative fluctuations.

How can we use this mechanism for low relative fluctuations in the context of our proposal? Recall from section 2 that the lattices of physical interest associated with quasiclassical states have links which are of the order of the Planck length. A 'macroscopic' lattice operator, $\hat{O}_L$, associated with a classical function O is typically the sum over 'N'microscopic operators $\hat{O}_{I_L}$. The index I_L typically ranges over all the plaquettes/cells in a macroscopic volume. Since the cells are of Planck size, N is very large. This raises the possibility of constructing states with $\frac{1}{\sqrt{N}}$ relative fluctuations in the measurement of $\hat{O}$. We indicate how this could happen below and show that it is possible to construct such states in the next section.

¿From (2) and (4) it is easy to see that

$$\hat{O}^2 = \sum_L \hat{P}_L \hat{O}_L \hat{P}_L \hat{O}_L \hat{P}_L. \tag{24}$$

It can be checked that

$$< \sum_L \hat{P}_L \hat{O}_L^2 \hat{P}_L > \geq < \hat{O}^2 > . \tag{25}$$

$$\Rightarrow (\Delta'\hat{O})^2 :=< \sum_L \hat{P}_L \hat{O}_L^2 \hat{P}_L > - < O >^2 \geq (\Delta\hat{O})^2. \tag{26}$$

It can be verified that $\Delta'\hat{O}$ evaluated in the quasi-classical state based on the lattice L_0 is given by

$$\Delta'\hat{O} = \Delta\hat{O}_{L_0}. \tag{27}$$

But

$$\hat{O}_{L_0} = \sum_{I_{L_0}=1}^{N} \hat{O}_{I_{L_0}}. \tag{28}$$

Then, if the $\hat{O}_{I_{L_0}}$ are sufficiently uncorrelated in the state, we have for $I_{L_0} \neq J_{L_0}$ that

$$< \hat{O}_{I_{L_0}} \hat{O}_{J_{L_0}} > \approx < \hat{O}_{I_{L_0}} >< \hat{O}_{J_{L_0}} > . \tag{29}$$

Then (28) implies that

$$(\Delta \hat{O}_{L_0})^2 \approx \sum_{I_{L_0}=1}^{N} (\Delta \hat{O}_{I_{L_0}})^2.$$ (30)

Typically, we expect $< \hat{O}_{I_{L_0}} >$ and $\Delta \hat{O}_{I_{L_0}}$ to be of order 1 times some microscopic (in general, dimension-full) constant and $< \hat{O} >= \sum_{I_{L_0}=1}^{N} < \hat{O}_{I_{L_0}} >$ to be of order N times the same constant [8]. Then we get

$$\frac{\Delta \hat{O}}{< \hat{O} >} \leq \frac{\Delta' \hat{O}}{< \hat{O} >} = \frac{\Delta \hat{O}_{L_0}}{< \hat{O}_{L_0} >} \approx \frac{1}{\sqrt{N}}.$$ (31)

In section 4 we shall examine some classical functions and their flux-based lattice approximants, and apply the strategy of this section to construct states with low relative fluctuations of the corresponding operators.

4 .Kinematical $2 + 1$ gravity

In subsections 4.1-4.3, we explore our ideas in the context of 2 spatial dimensions. In 4.1 we define some macroscopic functions and construct their quantum analogs in accordance with (4). In 4.2 we construct candidate quasi-classical states. In 4.3 we show that the relative fluctuations of the macroscopic operators defined in 4.1 can go as $\frac{1}{\sqrt{N}}$ in accordance with the ideas of section 3.

Unfortunately, for the reason mentioned in footnote 8, it is difficult to keep the scale of the fluctuations of these operators smaller than the typical scale of the corresponding classical quantities. Hence, not all of our ideas are successfully implemented. A discussion presenting ways in which our states may be modified, or our strategy refined is also contained in section 5.

In subsection 4.4 we indicate how to generalize our constructions to three spatial dimensions. We note that the same difficulties with the scale of the fluctuations arise there, too, and hence our construction of quasi-classical states is not yet satisfactory.

4.1 The macroscopic observables

In two spatial dimensions the phase space variables are a densitized triad and a $SU(2)$ connection $(\tilde{E}_i^a, A_a^i)$ [23] where i is an $SU(2)$ Lie algebra index and a is the spatial index. The metric is constructed from $\tilde{E}_i^a$ through $\tilde{E}_i^a \tilde{E}^{bi} = q q^{ab}$. In two dimensions the spatial geometry is determined if the lengths of all curves in the 2-manifold are specified. Moreover, for non-degenerate $\tilde{E}_i^a$ (i.e. for $\tilde{E}_i^a$ which define non-degenerate 2 metrics), the information in the curvature F_{ab}^i of the connection is coded in the local expressions $\tilde{E}_i^a F_{ab}^i$ and $\epsilon_k^{ij} \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k$. Hence the classical functions of interest are the length of an arbitrary curve 'c', $l(c)$, the 'vector constraint', $D(\vec{N}) = \int N^a \tilde{E}_i^a F_{ab}^i$,

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[8]Unfortunately, as we shall see in section 5, our strategy is not entirely successful because this expectation is not quite true for the operators and the states that we examine in section 4.

and the 'scalar constraint', $S(N) = \int \underset{\sim}{N} \epsilon^{ij}_k \tilde{E}^a_i \tilde{E}^b_j F^k_{ab}$ where N^a is an arbitrary vector field and $\underset{\sim}{N}$ is a density -1 scalar.

The corresponding operators are constructed as follows. The length operator can be constructed independent of the strategy of section 3, in the same fashion as the area operator in 3d. The eigenstates of the length operator $\hat{l}(c)$ are the spin network states and their eigen values have a contribution of $\lambda_j = 2l_P\sqrt{j(j+1)}$ for every intersection of the curve c and a link of the spin network colored by j. Here $l_P := \iota l_{0P}$. Note that in the language of section 3, this operator induces length operators $\hat{l}_L(c)$ in any lattice L.

The two sets of connection dependent operators can be defined first on a lattice L and then promoted to genuine operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_H)$ through (4).

$$\hat{D}_L(\vec{N}) = \frac{1}{4} \sum_{v,p=l_{p1} \wedge l_{p2} \to v} \hat{F}(p) \cdot (\hat{E}(v, l_{p1})N(v, l_{p2}) - \hat{E}(v, l_{p2})N(v, l_{p1})) + \text{H.T.} \quad (32)$$

where the sum runs over all vertices and all plaquettes that contain each given vertex (at vertex v the orientation of plaquette p is given by an ordered pair of links $l_{p1} \wedge l_{p2}$), $\hat{F}^i(p) = \frac{-i}{2}\text{Tr}(H(p)\sigma_i)$ ($H(p)$ is the holonomy around plaquette p) and $\hat{E}(v, l)$ acts as a left invariant vector field (multiplied by a factor of l_P) on functions depending on the holonomy along the link l oriented away from vertex v . 'H.T.' refers to Hermitian transpose.

$\hat{E}(v, l)$ can be interpreted as the triad operator smeared over a line transverse to the link l, but not crossing l in the center but at v. $\hat{F}(p)$ contains the information of the curvature smeared in the plaquette (plus higher order terms in the curvature that are not small in general). Thus, $\hat{E}$ and $\hat{F}$ are related to the triad and the curvature times factors of the lattice spacing a_g, measured by the macroscopic metric induced by the length operator in our state. Equation (32) provides a discretization of the classical vector constraints if the vector field N^a and the collection of weights assigned to the lattice links are related by $N(v, l_{p1})\hat{l}_{p1} + N(v, l_{p2})\hat{l}_{p2} = \frac{1}{a_g}\vec{N}(v)$, with $\hat{l}_{p1}, \hat{l}_{p2}$ being unit vectors in the direction of two of the links starting at v and forming a right-handed basis.[9] A definition of $\hat{D}_L(\vec{N})$ which corresponds to the classical function $D(\vec{N})$ for states with arbitrary valence would be more cumbersome to write. Since most of the vertices in the states that we will construct are four valent the expression (32) for $\hat{D}_L(\vec{N})$ is good enough for our purposes.

The other family of operators is defined by

$$\hat{S}_L(\underset{\sim}{N}) = \frac{1}{4} \sum_{v,p=l_{p1} \wedge l_{p2} \to v} \underset{\sim}{N}_L(v)\hat{F}^i(p)\hat{E}^j(v, l_{p1})\hat{E}^k(v, l_{p2})\varepsilon_{ijk} + \text{H.T.} \quad (33)$$

For this family of operators, the expectation values (on states with mostly four valent vertices) will approximate the classical functions known as the scalar constraints if the scalar of density weight -1 labeling the functions is related to the collection of weights assigned to the vertices by the relation $\underset{\sim}{N}_L(v) = \frac{1}{a_g^2}\underset{\sim}{N}(v)$.

---

[9]We assume that the vertex is four valent and formed by the intersection of two smooth curves; in this way, the definition does not depend on the choice of links to form the basis.

4.2 States with $\frac{1}{\sqrt{N}}$ relative fluctuations

In this section we display candidate quasi-classical states which provide a realization of our idea of $\frac{1}{\sqrt{N}}$ relative fluctuations. As mentioned earlier and discussed in section 5, the states which we construct are not completely satisfactory quasiclassical states. Neverthless, we present the construction of the candidate quasiclassical states in detail in the hope that this may fuel future efforts towards modifying our present strategy appropriately.

We shall display candidate quasi-classical states approximating homogeneous geometries and connections. This family of states includes, for example, states that generate expectation values approximating Euclidean metrics and flat connections on a torus, as well as states which approximate round spheres with constant curvature $SU(2)$ connections on them[10].

To make the macroscopic geometry (locally) isotropic, the physical lattice prescribed by the state will cover space with domains with the connectivity of a regular square lattice; these domains will be separated by narrow bands. We demand the distribution of orientations of the regular domains to be isotropic. The dominant contributions to any macroscopic observable will be those coming from the interior of the regular domains, and many domains will be involved in any macroscopic measurement. Thus, macroscopic observables will lose track of the connectivity of the lattice which will only be obvious at the micro-scale.

Our lattice should be composed by regular domains of typical size $D >>> l_P$ and have a linear density of links $\rho_l = \frac{k}{l_P}$. This is the density of intersections of the lattice links with any curve which wiggles only at the macroscopic scale (technically, its radius of curvature should be macroscopic). With this linear density of links the density of plaquettes is $\rho_p = (\frac{k\pi}{4l_P})^2$. We will later show that $k = \frac{2}{\sqrt{3}}$ is the correct value of this parameter, given the form of the states described below.

The states will be constructed as products; to each regular domain we will assign a factor and a separate factor will be assigned to the region between the regular domains. The factors assigned to regular domains will also be constructed as products. Taking advantage of the regularity, the interior of the domains are divided into black and white plaquettes in an alternate fashion. In the chess-board-like geometry of the interior of the domains we will assign factors of the wave function only to the black plaquettes asking that the color $n = 2j = 0$ does not appear in the spin network decomposition of any of the factors. Due to the alternate plaquette geometry, the color assigned to the links in a spin network decomposition of the state would be exactly the one coming from its only black plaquette neighbor. In this way our quasi-classical state will provide a physical lattice. It will be important that the spin network decomposition of the state does not acquire any zero color in the region between the regular domains to make sure that the state does prescribe a physical lattice and not a collection of separate domains. A technique to fit the domains together will be described after the contributions from the interior of the domains are

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[10] We remind our reader that the macroscopic observables that we are studying now are of local character and therefore two gauge inequivalent classical flat connections would appear indistinguishable to our "magnetic flux type" observables.

explained.

As we mentioned earlier, the factor of the wave function assigned to a domain is a product of factors associated to plaquettes

$$\psi_D = \prod_{p \in Bl} \psi_p \qquad (34)$$

where Bl contains alternate plaquettes. Since there are many more plaquettes in the interior of the domains than in the region between domains, to approximate any macroscopic observable we need to adjust only the factors associated to interior plaquettes. Furthermore, since we will illustrate our construction with a state approximating a homogeneous geometry and a homogeneous connection, all the factors ψ_p from the interior plaquettes can be taken equal. We choose

$$\psi_p = \cos\theta \phi_{n=1} + \sin\theta \phi_{n=2} \quad . \qquad (35)$$

where $\phi_i(A)$ is the trace of the holonomy around plaquette p in the spin $\frac{i}{2}$ representation. Other choices of ψ_p are possible; we chose the simplest states that defined a physical lattice and had small spins.

Let us now describe the assignment of factors of the wave function to the regions of the lattice that do not belong to the domains described earlier.

To simplify our work we will restrict the geometry of the lattices that we consider. First, we concentrate on the boundary of the regular domains. The boundary of the regular domains will be composed only of black plaquettes (one may construct these kind of geometries by erasing the boundary links of the white plaquettes in the boundary). In addition, we will only consider geometries where the black plaquettes in the boundary of the regular domains share at least two vertices with the black plaquettes in the domain, and if one of these black plaquettes shares only two vertices with the interior plaquettes this plaquette must be triangular (the plaquettes in the interior of the domains are all square plaquettes, but in the boundary we allow also triangular plaquettes). In these geometries it follows that all the plaquettes having a link in the boundary are black and that these boundary plaquettes have at most one vertex that is not shared by any plaquette in the regular domain. We will call these vertices black vertices. Apart from these vertices, in the boundary of the regular domains, there are vertices that are shared by interior plaquettes. We will call these vertices, white vertices.

We will take these boundary vertices as data and construct the rest of the lattice by filling the gaps in between the domains in a way that lets us assign a simple factor of the wave function to this "in between" region of the lattice.

At a bigger scale we can use the regular domains as cells of a latticization [11] of the surface Σ. Neighboring domains are separated by bands (analog of links) and these bands meet in rotaries (analog of vertices). For convenience, in the lattices that we will consider the bands and rotaries will have no internal vertices, and the rotaries will have no internal links. In other words, the rotaries are simply cells whose links are boundary links of the bands or boundary links of the regular domains. On the other

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[11] The 1-skeleton of cellular complex with the topology of Σ.

hand, the bands have interior links, but the interior links of each band are restricted to form a closed curve γ_B joining black vertices (either joining black vertices from the same domain or joining black vertices of neighboring domains). See the figure.
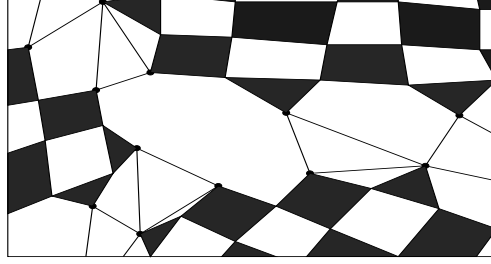

Fig. 1 We show a region of the lattice that is in the boundary between three regular domains. The domains have square lattice connectivity in their interiors and we assign factors of the wave function to the black plaquettes in the chess-board geometry of the interiors. Separate factors are assigned to each of the bands that serves as boundary between two regular domains; these factors are spin networks of color one whose graph γ_B is drawn joining the black vertices of the figure without retracing any line.

Due to the connectivity of its interior links, we can assign to a band a factor of the wave function which is simply the spin network determined by its graph and the color $n = 1$, $\psi_B = \psi_{\gamma_B, n=1}$. Since the links of a band join only black vertices, when we multiply the band factors with the domain factors, the spin network decomposition of the state will not have color zero in any link. That is, our proposed wave function

$$\psi = [\prod_D \psi_D][\prod_B \psi_B] \tag{36}$$

defines a physical lattice which encodes the topology of Σ.

It is clear that a space manifold with arbitrary topology can be covered by a lattice composed by disconnected domains with trivial topology (whose interiors have the connectivity of a regular square lattice and required link density) and joined by narrow band regions where the lattice does not need to posses any regularity. Thus, from the classical data of a Euclidean torus with a flat $SU(2)$ connection we can construct our candidate quasi-classical states based on the required lattice, and analogously from the classical data of a round sphere with a constant curvature connection we can construct candidate quasi-classical states.

4.3 Expectation values, fluctuations and correspondence

Given a macroscopic curve we want to calculate the expectation value of its length $< \hat{l}_L(c) >$. The calculation is easy. In two dimensions the eigenstates of the length operator are spin network states and their eigen values have a contribution of $\lambda_n = \frac{1}{2} l_P \sqrt{n(n+2)}$ for every intersection with the curve. According to our conventions, the total number of intersections is $\frac{k}{l_P} l_g(c)$, where $l_g(c)$ is the length of the curve. Now we

will make two approximations; we will assume that every plaquette which intersects c intersects it twice and that the parameter θ is small (because, as we will see, it is linked to the contribution of one plaquette to the curvature of the connection). In this way we get

$$
\begin{aligned}
< \hat{l}_L(c) > &= 2 \sum \lambda_{n=1} \cos^2 \theta + \lambda_{n=2} \sin^2 \theta \\
&\approx \lambda_{n=1} \frac{k}{l_P} l_g(c)
\end{aligned}
\tag{37}
$$

where the sum runs over all plaquettes that intersect the curve c. ¿From this formula we deduce $k = \frac{l_P}{\lambda_{n=1}} = \frac{2}{\sqrt{3}}$.

Now we calculate the expectation values of the connection related observables.

The connection measuring operators are a sum of terms. The dominant contributions to the expectation values come from the interior of the regular domains. Since each term in the sum only affects a few factors in the wave function, the contributions can be easily calculated. Since we deal only with homogeneous wave functions, the calculations simplify even more. If we had allowed the parameter θ to be a function of the plaquette the basic mechanism of our proposal would still work; we would be able to approximate many more classical configurations, but the calculations would not be as simple.

It turns out that the alternate plaquette nature of the wave function makes $< \hat{D}(\vec{N}) >= 0$ for all constant shifts. This would be true even if the parameter θ were a function of the plaquette. To see this, it is convenient to rewrite the action of the vector fields $\hat{E}^i(v, l_x)$ in a way that is tailored to act on wave functions that are products of plaquette factors. When acting on fucntions of type (34), $\hat{E}^i(v, l_x) = \frac{il_P}{2}[\hat{L}^i(p) - \hat{L}^i(p, -\hat{y})]$, where $\hat{L}^i(p), \hat{L}^i(p, -\hat{y})$ are the left invariant vector fields acting on functions of the group assigned to the plaquette (p) and the neighbor of (p) in the $(-\hat{y})$ direction respectively. Due to the alternate plaquette geometry, the only terms that do not vanish in the expectation value are the ones containing the factor $\hat{F}(p) \cdot \hat{L}(p)$. The result is

$$
< \hat{D}(\vec{N}) >= \sum_{p \in Bl} C_0 \mathrm{Div} N(p)
\tag{38}
$$

where, in the plaquette (p) defined by the vertices $(v, v + l_{p1}, v + l_{p1} + l_{p2}, v + l_{p2})$,

$$
\mathrm{Div} N(p) = N(v, l_{p1}) + N(v, l_{p2}) - N(v + l_{p_1}, l_{p1}) + N(v + l_{p_1}, l_{p2}) -
$$
$$
N(v + l_{p_1} + l_{p2}, l_{p1}) - N(v + l_{p_1} + l_{p2}, l_{p2}) + N(v + l_{p2}, l_{p1}) - N(v + l_{p2}, l_{p2})
$$

and C_0 is calculated for any interior plaquette (p) as

$$
C_0 = \frac{il_P}{2} < \hat{F}(p) \cdot \hat{L}(p) - H.T. >
\tag{39}
$$

For small θ, $C_0 \approx -\frac{5}{4}\theta l_P$.

For similar reasons, the expectation value of $\hat{S}(\underset{\sim}{N})$ simplifies greatly and we get

$$
< \hat{S}(\underset{\sim}{N}) >= \sum_{p \in Bl} \underset{\sim}{N}_L(p) C_0
\tag{40}
$$

We now discuss the fluctuations. For the length operator one can consider $\hat{l}_L(c) = \sum_p \hat{l}_L(c_p)$. Here, the sum is over black plaquettes which intersect the curve 'c' and c_p refers to the segment of c in the black plaquette p. Then, one can easily check that

$$\Delta \hat{l}_L(c) \approx .6 l_P \cos(\theta) \sin(\theta) \sqrt{N} \qquad (41)$$

where N is the number of black plaquettes which intersect the curve. For small θ, $\Delta \hat{l}_L(c) \approx .6 l_P \theta \sqrt{N}$. This is consistent with the fact that for $\theta = 0$ our states are eigen states of the length operator.

In the case of the vector constraint operator and the scalar constraint operator the calculations are not as simple and multiple contributions appear. Nonetheless, due to the nature of our state, in the regular domains a plaquette is significantly correlated with only a few nearby plaquettes. The boundaries of the regular domains contribute only a small amount to $(\Delta \hat{D}(\vec{N}))^2$ and hence it is easy to verify that $(\Delta \hat{D}(\vec{N}))^2$ is proportional to the number of plaquettes 'N' in the regular domains. Similar considerations apply to $(\Delta \hat{S}(\underset{\sim}{N}))^2$. Thus our idea of $\sqrt{N}$ fluctuations is successfully implemented in the states we have displayed.

One difference (in detail) from the calculation of length fluctuations is that $\Delta \hat{D}(\vec{N})$ and $\Delta \hat{S}(\underset{\sim}{N})$ do not vanish when $\theta = 0$. For example, an important contribution to $(\Delta \hat{D}(\vec{N}))^2$ comes from terms of the form $F(p) \cdot L(p) l_P$; we get (to second order in θ)

$$(\Delta F(p) \cdot L(p) l_P) \approx [\frac{7}{8} - \frac{7}{16}\theta^2] l_P \qquad (42)$$

This is when (p) is a black plaquette; for white plaquettes we get

$$(\Delta F(p) \cdot L(p) l_P) \approx O(l_P) \qquad (43)$$

regardless of θ.

The correspondence between commutator expectation values and Poisson brackets is a more involved calculation and we have not investigated this in any detail. Therefore, we restrict ourselves to the following remarks.

In the case of the length operators it is easy to see that

$$< [l_L(c), l_L(c')] >= 0. \qquad (44)$$

In three spatial dimensions there are general reasons to expect that, in quasi-classical states, the expectation value of the commutator of the area operators and its fluctuations are small [24]; the argument also applies to the length operators in the two dimensional case. With regard to the calculations for $\hat{D}(\vec{N})$ and $\hat{S}(\underset{\sim}{N})$, the results of [25, 26] applied to the present context support the correspondence between Poisson brackets and commutator expectation values in quasiclassical states.

4.4 Extension to 3+1 dimensions

There is a natural analog of the set of observables that determine our quasi-classicality criterion. For the geometry, the area operators and for the connection we could

consider the induced connection on surfaces with arbitrary embedding and measure the connection with the same type of "magnetic flux type" operators (that would have the smearing surface as an extra label). This set of operators seems to be large enough and would be very close to the 2D case studied here. However, we have not done any serious study of its properties. Other families may prove to be better.

Our family of candidate quasi-classical states is tightly tied to a two-dimensional space. However, the main idea is easily extendible to other dimensions. Now we describe it briefly.

The three-dimensional chess-board geometry inside the regular domains is such that black cubes meet only at their vertices. At a vertex (v_0) two opposite octants are black and the rest are white; one can color the whole lattice translating the painted cubes meeting at v_0 in the three cartesian directions by an even number of steps. To each black cell we assign the factor $\psi = \cos\theta\psi_2 + \sin\theta\psi_4$ with ψ_{2n} being the spin network state with color $2n$ in the edges of the black cube. (Other choices with smaller colors are also possible.)

The factors assigned to the bands in the two dimensional case were found using a procedure that can be adapted to the three-dimensional case. We require that at the boundary of the regular domains the black cells share at least three vertices with the other black cells in the domain. Then we change the shape of the boundary black cells to have only one free vertex. These free vertices are the black vertices needed to construct the lattice in the band region and assign a factor of the wave function to each band. We use these factors that tie neighboring domains with a single spin network of color one per band.

In this way we construct a family of states each of which defines a physical lattice. By adjusting the multitude of free parameters (density of intersections of the lattice links with surfaces that look flat at the microscale and θ as a function of the cells of the regular domains) we should be able to approximate any given classical data. Also, we can restrict to homogeneous states that we would only be able to approximate homogeneous classical data.

5. Discussion

It is important to clarify that our intent is *not* to provide an alternative quantization to that of loop quantum gravity. Loop quantum gravity is a theory still under construction and thus, a yet incomplete enterprise. Even at the kinematic level, as we have argued in section 2, the theory is incomplete in that its most straightforward interpretation does not lead us to the classical limit. We view this work as an attempt to remedy this particular instance of incompleteness by providing a framework to discuss quasiclassicality. As we have stressed before, this framework is applicable only to the calculation of expectation values and fluctuations in quasiclassical states. Thus, we do not yet understand the *transition* from the fully quantum regime to the regime in which our framework is proposed to apply, namely the semiclassical regime. However, it is clearly necessary to establish *some* framework which defines a satisfactory notion of quasiclassicality, before the study of this transition can be undertaken

and therein, we believe, lies the virtue of this work.

After these preliminary remarks, we discuss the following issues which arise in the context of our proposal.

(i) Superselection sectors: Given a quasiclassical state associated with a lattice 'L', it is clear that no operator of the form (4) maps the state out of the space of spin networks based on L. Thus, if operators of the form (4) were the only operators in loop quantum gravity, we would be faced with the existence of uncountably many superselected sectors, one for every choice of lattice. However, as stressed in our preliminary remarks above, operators of the form (4) have been constructed solely to probe the classical limit in terms of their expectation values and fluctuations in quasiclassical states. There exist for example, in addition to such operators, microscopic (Planck scale) operators in loop quantum gravity which need not be of the form (4). Such operators can map quasiclassical states out of the putative superselected sectors.

As mentioned earlier in this section, we do not yet understand the relation between the fully quantum regime and the semiclassical regime as defined through our proposal. Hence we do not know the role of these Planck scale operators in the semiclassical regime. Thus, even if there are superselected sectors at the kinematic level, these sectors may disappear when the dynamical aspects of quantum gravity at the Planck scale are incorporated.

(ii) Ambiguities in the construction of the operator $\hat{O}$ corresponding to the classical quantity O: On a fixed lattice 'L' , there are (infinitely) many microscopically distinct lattice approximations to the same continuum quantity. Thus, there are infinitely many, distinct ways to construct $\hat{O}$ through (4). It is not clear if we should demand that our state be quasi-classical with respect to *all* possible choices of $\hat{O}$, and if so, whether there exist any such states.

(iii) The algebra of operators of the type $\hat{O}$: A qualitatively different ambiguity results from an examination of the algebra of operators of the type $\hat{O}$. Let the quasi-classical state of interest be associated with the lattice L. Consider the operators $\hat{A}$ and $\hat{B}$ constructed from A_L and B_L through (4). For simplicity, assume $[\hat{A}, \hat{B}] = 0 = [\hat{A}_L, \hat{B}_L]$. Then the operator corresponding to the quantity AB can be constructed either as

$$\hat{AB} = \sum_L P_L \hat{A}_L \hat{B}_L P_L \tag{45}$$

or as

$$\hat{A}\hat{B} = \sum_L P_L \hat{A}_L P_L \hat{B}_L P_L. \tag{46}$$

Since $\hat{AB} \neq \hat{A}\hat{B}$, there is an ambiguity in the definition of the operator corresponding to AB.

There is a special case in which this ambiguity is irrelevant. If the quasiclassical state based on a lattice is such that $\Delta'A$, $\Delta'B$ (see equation (26)) are small compared

to $< A >, < B >$ then it can be shown that the uncertainty principle implies

$$\frac{< \hat{A}\hat{B} >}{| < \hat{A} >< \hat{B} > |} = \frac{< \hat{A}\hat{B} >}{| < \hat{A} >< \hat{B} > |} + \epsilon, \tag{47}$$

where

$$|\epsilon| \leq \frac{\Delta'\hat{A}}{| < \hat{A} > |} \left(1 + (\frac{\Delta'\hat{B}}{| < \hat{B} > |})^2\right)^{\frac{1}{2}}. \tag{48}$$

Thus, in this special case, this particular ambiguity in the definition of the operator corresponding to AB is of no consequence.

(iv) How small is the microscopic 'magnetic flux'?: Our construction of states with small fluctuations is based on the premise that every macroscopic quantity is N times some microscopic quantity with N very large. Therefore, it is essential that the characteristic scale of the microscopic quantity be much smaller than that of the macroscopic quantity. In this regard, the 'magnetic' flux presents the following dilemma. [12]

The classical 'magnetic' field is related to the spatial and extrinsic curvatures through

$$F_{ab}^i = R_{ab}^i + 2\iota D_{[a}K_{b]}^i + \iota^2 \epsilon_{jk}^i K_a^i K_b^j, \tag{49}$$

where D_a is the operator compatible with the triad and R_{ab}^i is its curvature. K_a^i is closely connected with the extrinsic curvature when all the constraints of general relativity are imposed. If the Immirzi parameter, ι, is of order unity, then in any physically reasonable coordinates, it is clear that the classical scale for F_{ab}^i is much smaller than an inverse Planck area. Hence, the magnetic flux through a plaquette of Planck size should be much less than unity. The microscopic flux operator for a Planck size plaquette 'p' of the lattice associated with a quasiclassical state is $Tr\hat{H}_p\tau^i$. This operator 'lives' on the copy of $SU(2)$ associated with 'p' and clearly its fluctuations are of order unity for the type of state contemplated in section 4. This translates to *huge* fluctuations of order inverse Planck area in the associated microscopic magnetic field. Hence the microscopic field fluctuation is much larger than the macroscopic scale and our ideas do not apply. It can be seen that such large fluctuations in the microscopic field, for our states, result in large fluctuations in the macroscopic field. For the macroscopic field averaged over a surface of macroscopic area A, the fluctuations turn out to be of the order $\frac{1}{\sqrt{A}l_P}$ where l_P is the Planck length (see equation (50)). Thus, the macroscopic fluctuations swamp out typical classical values!

Nevertheless, let us see how far we can push our ideas. We need to somehow magnify the typical macroscopic scale. Notice that this is possible (from (49)) if we choose a large value of ι. Then small fluctuations of the extrinsic curvature magnify to large fluctuations of the magnetic field/flux. Thus, it is possible to salvage our ideas

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[12]Although our arguments involve quantities which are not $SU(2)$ gauge invariant, it is easy to see that our conclusions apply to any gauge invariant quantities constructed from the magnetic field such as D and S of the previous section

by appealing to a large ι. However, in such a case, it is not clear that the curvature can be identified with the plaquette holonomy since this identification assumes that the plaquette flux is small. Nevertheless, if we ignore this objection and choose ι to be large and if we still identify ιl_{0P}^2 (in 3d) with the Planck area, l_P^2, then it is clear that G_0 cannot take the value of Newton's constant but must be interpreted as a bare constant.

Another way to improve matters, say in 3d, is to decrease the effective 'magnetic field' by increasing the plaquette size. This is possible if the quasiclassical state is defined by high spins so that the area of a plaquette is of the order of $l_{typical}^2 = j_{typical} l_P^2$. Here $j_{typical}$ characterizes the scale of the (high) spins. Note that fluctuations in area will then be characterized by $l_{typical}^2$ rather than l_P^2; hence $l_{typical}$ must be much less than the macroscopic scale.

For example, in 3d, this idea applied to the (non gauge invariant) magnetic flux, $\Phi^i(S)$ (see section 3), through a surface S of area A results in the following estimates. Let N be the number of plaquettes tiling S. Then we have $A = N l_{typical}^2$ and $\Delta\Phi^i(S) \sim \sqrt{N}$. Then the fluctuation in the average magnetic field $B^i = \frac{\Phi^i(S)}{A}$ is

$$\Delta B^i := \frac{\Delta\Phi^i(S)}{A} \sim \frac{1}{\sqrt{A l_{typical}^2}} \tag{50}$$

instead of $\frac{1}{\sqrt{A l_P^2}}$.

The emergence of a scale defined by the quasiclassical state between the macroscopic scale and the Planck scale can be argued, independently of our specific "$\frac{1}{\sqrt{N}}$" inspired constructions. The area operators (length in 2d) are the fundamental metric dependent operators. The uncertainities in the measurement of connection dependent operators are constrained through the uncertainity principle by the size of the fluctuations in the area (length) and the commutator between the area and the connection dependent operators. The larger the permissible uncertainity in the area, the smaller is the achievable uncertainity in the connection operators. The scale of area fluctuations defines $l_{typical}$ and a characteristic 'spin', $j_{typical} := (\frac{l_{typical}}{l_P})^2$. Clearly, $j_{typical}$ must characterize the scale of spins occurring in a spin network decomposition of the quasiclassical state. As mentioned earlier, for smoothness of the macroscopic geometry, $l_{typical}$ must be much less than the macroscopic scale.

A similar picture of the classical limit arises in quantum Regge calculus. The relation between the Ponzano-Regge-Turaev-Viro partition function and the Regge action for three dimensional Euclidean spacetimes holds in the large j limit [27, 28]. This means that classical smooth spacetimes have origin in states whose quantum geometry defines a scale $j_{typical} l_P$ which is macroscopically small (to approximate a smooth geometry at macroscopic scales) and at the same time is much bigger than the Planck scale.

(v) The possibility of incorporating spatial diffeomorphism invariance into our proposal: Since our constructions do not use any external fixed structures, they are covariant with respect to spatial diffeomorphisms. Hence they ought to generalise to

a spatially diffeomorphism invariant setting. Such a setting is provided by the Rovelli model [8] which combines the Hussain-Kuchař model [29] with a matter reference system. In the context of our constructions, the lattice associated with a quasiclassical state for the classical data $(\tilde{E}_0^a, A_{0a})$ can be specified through the choice of a particular eigenstate of the fermion fields in the Rovelli model. The fermion fields define surfaces and the cells of the lattice can be located through the intersections of these surfaces. Let us refer to the eigenstate of the fermion fields which specifies a lattice L as $|L_F>$. In the Rovelli model, it is possible to construct classical diffeomorphism invariant 'gravitational' quantities by involving the reference matter fields in their definition. Our proposal would indicate that an analysis of the classical limit for such diffeomorphism invariant configurations of the 'gravitational' field and the matter reference system, can be done in terms of diffeomorphism invariant operators of the form

$$\hat{O} := \sum_L P_{L_F} P_L \hat{O}_L P_L P_{L_F} \tag{51}$$

Here O_L is the lattice approximant of the diffeomorphism invariant classical quantity O and $P_{L_F} = |L_F><L_F|$ is the projector onto the 'reference system lattice'. The subsequent considerations of section 3 can be also be suitably generalised to the Rovelli model. The quasiclassical state thus constructed will be one for the 'gravitational' variables only- the matter variables are still very quantum because the 'matter part of the state', $|L_F>$, is an eigenstate of the matter fields.

6. Conclusions

In this work we have shown that there are no quantum states in the kinematical Hilbert space of loop quantum gravity which approximate, in expectation value, classical holonomies along all possible loops and that, at best, it may be possible to approximate only a countable number of classical holonomies. Since the holonomy variables are the primary variables of the loop approach, a new framework to analyse the classical limit of kinematic loop quantum gravity is needed which takes into account the above result.

Next, we have proposed a framework in which the choice of a countable number of loops is made without breaking spatial covariance by identifying the loops with those which are contained in the graph underlying the quasiclassical state itself. Since the graph is required to be a lattice we are able to import techniques from lattice gauge theory to examine various operators of interest (see section 2). This part of our work is quite robust.

Next, inspired by the mechanism for low relative fluctuations in statistical mechanics, we explicitly constructed candidate quasiclassical states in 2 spatial dimensions. Although we could successfully implement this mechanism for low relative fluctuations, the states were not completely satisfactory because the fluctuations (as opposed to the relative fluctuations) were very large. More precisely, we were able to construct states which had fluctuations of order $\sqrt{N}$ times some naturally occurring microscopic unit, with N large. Under the assumption that typical classical values

were of order N times this unit, these states had $\frac{1}{\sqrt{N}}$ relative fluctuations. However, on closer examination we found that this assumption was unwarranted and that the microscopic unit was not small enough. As a result the fluctuations swamped out typical classical values. Nevertheless, the fluctuations were reduced drastically in size as compared to the fluctuations at the Planck scale. For example, at the Planck scale, curvature fluctuations are expected to be of the order of the inverse Planck area; the mechanism of low relative fluctuations reduced the fluctuations in the macroscopic curvature by a factor of $\frac{l_P}{\sqrt{A}}$ where A is the macroscopic area (see **(iv)**, section 5).

Even if our particular explicit construction of candidate quasiclassical states is irrelevant, it is still true that our proposal establishes a connection with lattice gauge theory and reinforces the 'weave' based picture of discrete space. The very fact that we have made a connection to lattice gauge theory techniques raises the issue of 'bareness' of the gravitational coupling and the possible existence of several phases and length scales in our quantum theory. In lattice gauge theory, the coupling is renormalized, and phases appear due to *dynamical* considerations. The considerations of the previous section point towards the need of considering scenarios for different phases and renormalization of coupling constants at the kinematic level. Certainly not much more can be inferred in the absence of dynamics, i.e., the construction of a projector into the space of physical states (in Hamiltonian language, the imposition of the diffeomorphism and, especially, the Hamiltonian constraint).

Since our proposal is new and unconventional, it is essential to confront our constructions with physically reasonable criteria and modify our proposal accordingly. We have attempted to do this to some extent in the previous section, but the consequences of our formalism need to be explored thoroughly before accepting it as a viable approach towards an analysis of the classical limit.

Nevertheless, given that a new framework is needed which identifies a countable set of loops, it seems inevitable that projectors onto this set of loops (such as we have defined) play a crucial role. This fact, along with the need to preserve spatial covariance and the requirement of hermiticity of the operator versions of real classical functions, naturally point towards our specific proposal.

Loop quantum gravity is a very conservative approach to the problem of quantum gravity in that it is an attempt to combine the principles of quantum mechanics with that of general relativity in accordance with tried and tested rules. We believe that the real virtue of the loop quantum gravity approach is that it captures, in a clear way, the points of tension between quantum mechanics and general relativity and hence suggests new ideas beyond the scope of its own framework, which may relax this tension.

In this respect, our work seems to emphasize structures *intrinsic* to the quantum states as important and hence points away from the embedded spin networks of Rovelli and Smolin [17, 18] to the intrinsically defined spin networks of Penrose [30]. In closing we note that the considerations of this work, the qualitative similarity of the resulting description of classical space with the quantum statistical mechanics description of a classical solid and considerations such as that of Jacobson [31], reinforce the idea that the dynamics of general relativity (and particularly the Hamiltonian constraint) may arise as a coarse grained/statistical description of fundamental degrees

of freedom at the Planck scale.

Appendix

A1

Let the space of loops with base point x_0 be $\mathcal{L}_{x_0}$. Denote the trivial loop by e. As in [32], consider the free vector space $\mathcal{FL}_{x_0}$ generated by loops in $\mathcal{L}_{x_0}$. On $\mathcal{FL}_{x_0}$, define the product law

$$(\sum_{i=1}^{n} a_i \alpha_i)(\sum_{j=1}^{m} b_j \beta_j) := (\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \alpha_i \circ \beta_j) \tag{52}$$

and the involution

$$(\sum_{i=1}^{n} a_i \alpha_i)^\dagger := \sum_{i=1}^{n} a_i^* \alpha_i^{-1}. \tag{53}$$

Here, a_i, b_j are complex numbers and $\alpha_i, \beta_j \in \mathcal{L}_{x_0}$.

Fix a connection A_{0a}^i and extend the definition of holonomy trace to $\mathcal{FL}_{x_0}$ by linearity so that

$$T_{\sum_{i=1}^{n} a_i \alpha_i}^0(A_0) = \sum_{i=1}^{n} a_i T_{\alpha_i}^0(A_0). \tag{54}$$

Next, define

$$I_{A_0} := \{\sum_{i=1}^{n} a_i \alpha_i \in \mathcal{FL}_{x_0} | \sum_{i=1}^{n} a_i T_{\alpha_i \circ \beta}^0(A_0) = 0 \text{ for every } \beta \in \mathcal{L}_{x_0}\}. \tag{55}$$

It can be checked that I_{A_0} is a two sided ideal in $\mathcal{FL}_{x_0}$.

Note that, since T^0 is an $SU(2)$ trace, under involution

$$T_{(\sum_{i=1}^{n} a_i \alpha_i)^\dagger}^0(A_0) = \sum_{i=1}^{n} a_i^* T_{\alpha_i}^0(A_0) = (\sum_{i=1}^{n} a_i T_{\alpha_i}^0(A_0))^*. \tag{56}$$

We choose A_0 such that there exists some loop $\tau \in \mathcal{L}_{x_0}$ for which

$$|T_\tau^0(A_0)| \neq 1. \tag{57}$$

Define the complex numbers $l_1(\tau), l_2(\tau)$ as

$$l_1(\tau) := T_\tau^0(A_0) + i(1 - (T_\tau^0(A_0))^2)^{\frac{1}{2}}, \tag{58}$$

$$l_2(\tau) := T_\tau^0(A_0) - i(1 - (T_\tau^0(A_0))^2)^{\frac{1}{2}} \tag{59}$$

and $\rho_1(\tau), \rho_2(\tau) \in \mathcal{FL}_{x_0}$ as

$$\rho_1(\tau) := (l_1(\tau) - l_2(\tau))^{-1}(\tau - l_2 e), \tag{60}$$

$$\rho_2(\tau) := (l_2(\tau) - l_1(\tau))^{-1}(\tau - l_1 e). \tag{61}$$

It can be checked that mod I_{A_0},

$$\rho_i(\tau)\rho_j(\tau) = \delta_{ij}\rho_i(\tau), \qquad \rho_1(\tau) + \rho_2(\tau) = e \tag{62}$$

and that for any $\alpha \in \mathcal{L}_{x_0}$

$$T^0_{\rho_1(\tau^{-1})\alpha}(A_0) = T^0_{\rho_2(\tau)\alpha}(A_0). \tag{63}$$
$$T^0_{(\rho_1(\tau))^\dagger \alpha}(A_0) = T^0_{\rho_1(\tau)\alpha}(A_0). \tag{64}$$

We shall further restrict attention to A^i_{0a} such that there exists some $a \in \mathcal{FL}_{x_0}$ for which

$$C := T^0_{\rho_1(\tau)a\rho_2(\tau)a^\dagger}(A_0) \neq 0. \tag{65}$$

Using the algebraic properties of the T^0 variables and (56), (62), (63) and (64) it can be verified that

$$U_\alpha(A_0) := \begin{pmatrix} 2T^0_{\alpha\rho_1(\tau)}(A_0) & \dfrac{2T^0_{\rho_1(\tau)\alpha\rho_2(\tau)a^\dagger}(A_0)}{\sqrt{2C}} \\ \dfrac{2T^0_{\rho_2(\tau)\alpha\rho_1(\tau)a}(A_0)}{\sqrt{2C}} & 2T^0_{\alpha\rho_2(\tau)}(A_0) \end{pmatrix} \tag{66}$$

is an $SU(2)$ matrix such that $U_\alpha(A_0)U_\beta(A_0) = U_{\alpha\circ\beta}(A_0)$ with $\frac{1}{2}TrU_\alpha(A_0) = T^0_\alpha(A_0)$. Details of this construction maybe found in [22].

We note that the proof of the above properties of $U_\alpha(A_0)$ depend solely on the *algebraic* properties of the T^0 (and their extensions to $\mathcal{FL}_{x_0}$) and the property (56) of the T^0 under involution; and is independent of the particular connection A_0. [13] These algebraic properties are shared by the $\hat{T}^0$ operators and the property (56) translates to adjointness properties of the $\hat{T}^0$ operators. Moreover, since these operators form a commutative algebra it can be verified that substituting $\hat{T}^0$ for all occurrences of $T^0(A_0)$ in the above construction, yields an $SU(2)$ valued operator $\hat{U}_\alpha$ such that $\hat{U}_\alpha\hat{U}_\beta = \hat{U}_{\alpha\circ\beta}$ and $\frac{1}{2}Tr\hat{U}_\alpha = \hat{T}^0_\alpha$. [14]

Now we can substitute $\hat{H}$ by $\hat{U}$ in the arguments of section 2 and obtain (17), this time, in a gauge invariant context with the (weak) restrictions (57) and (65) on the classical connection A^i_{0a}.

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[13] Provided, of course, that the various expressions in (66) are well defined. That they are indeed well-defined is guaranteed by the requirements (57) and (65).

[14] The counterpart of the restrictions (57) and (65) is the fact that some of the operators encountered (namely, $(\hat{l}_1(\tau) - \hat{l}_2(\tau))^{-1}$ and $\hat{C}^{-\frac{1}{2}}$) are unbounded. We assume that mathematical subtleties related to domain issues of unbounded operators can be taken care of in a more careful treatment.

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